Inequalities (Unit 1 – Unit 3)

Solutions to Exercises

Inequalities (Unit 1)

1. By the AM-GM inequality, we have $\frac{1+a_i}{2} \geq \sqrt{1 \cdot a_i}$, i.e. $1+a_i \geq 2\sqrt{a_i}$ for all *i*. Hence

$$\begin{aligned} 2^n &= (1+a_1)(1+a_2)\cdots(1+a_n) \\ &\geq \left(2\sqrt{a_1}\right)\left(2\sqrt{a_2}\right)\cdots\left(2\sqrt{a_n}\right) \\ &= 2^n\sqrt{a_1 a_2 \cdots a_n} \end{aligned}$$

Dividing both side by 2^n, we have $1 \geq \sqrt{a_1 a_2 \cdots a_n}$, so that $a_1 a_2 \cdots a_n \leq 1$.

2. By the AM-GM inequality, we have $\frac{1}{2}\left(\frac{a_1^{\ 2}}{a_2} + a_2\right) \geq \sqrt{\frac{a_1^{\ 2}}{a_2} \cdot a_2}$, i.e. $\frac{a_1^{\ 2}}{a_2} \geq 2a_1 - a_2$.

Similarly, we have $\frac{a_1^{\ 2}}{a_2} \geq 2a_1 - a_2$, $\frac{a_2^{\ 2}}{a_3} \geq 2a_2 - a_3$ and so on. Hence

$$\begin{aligned} \frac{a_1^{\ 2}}{a_2} + \frac{a_2^{\ 2}}{a_3} + \cdots + \frac{a_n^{\ 2}}{a_1} &\geq (2a_1 - a_2) + (2a_2 - a_3) + \cdots + (2a_n - a_1) \\ &= (2a_1 + 2a_2 + \cdots + 2a_n) - (a_2 + a_3 + \cdots + a_n + a_1) \\ &= a_1 + a_2 + \cdots + a_n \end{aligned}$$

<u>Alternative Solution</u>

Without loss of generality, assume $a_1 \geq a_2 \geq \cdots \geq a_n$. Then we have

$$a_1^{\ 2} \geq a_2^{\ 2} \geq \cdots \geq a_n^{\ 2} \quad \text{and} \quad \frac{1}{a_1} \leq \frac{1}{a_2} \leq \cdots \leq \frac{1}{a_n}.$$

Using the fact that Random Sum $\geq$ Reverse Sum, we have

$$\frac{a_1^{\ 2}}{a_2} + \frac{a_2^{\ 2}}{a_3} + \cdots + \frac{a_n^{\ 2}}{a_1} \geq \frac{a_1^{\ 2}}{a_1} + \frac{a_2^{\ 2}}{a_2} + \cdots + \frac{a_n^{\ 2}}{a_n} = a_1 + a_2 + \cdots + a_n.$$

3. Let $x = 1-a$, $y = 1-b$ and $z = 1-c$.

Then $a+b+c=2$ implies $a = 2-b-c = 2-(1-y)-(1-z) = y+z$.

Similarly, we have $b = z+x$ and $c = x+y$.

Hence the original inequality becomes $\frac{(x+y)(y+z)(z+x)}{xyz} \geq 8$, or $(x+y)(y+z)(z+x) \geq 8xyz$.

By the AM-GM inequality, we have $\frac{x+y}{2} \geq \sqrt{xy}$, i.e. $x+y \geq 2\sqrt{xy}$.

Similarly, $y+z \geq 2\sqrt{yz}$ and $z+x \geq 2\sqrt{zx}$.

Consequently, $(x+y)(y+z)(z+x) \geq \left(2\sqrt{xy}\right)\left(2\sqrt{yz}\right)\left(2\sqrt{zx}\right) = 8xyz$, completing the proof.

4. Without loss of generality, assume $a \geq b \geq c$. Then

$$\frac{1}{b+c} \geq \frac{1}{a+c} \geq \frac{1}{a+b}.$$

Using the fact that Direct Sum $\geq$ Random Sum, we have

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{b}{b+c} + \frac{c}{a+c} + \frac{a}{a+b}$$

Taking another random sum, we have

$$\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{c}{b+c} + \frac{a}{a+c} + \frac{b}{a+b}.$$

Adding the above two inequalities, we have

$$2\left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}\right) \geq \frac{b+c}{b+c} + \frac{a+c}{a+c} + \frac{a+b}{a+b} = 3,$$

so that $\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b} \geq \frac{3}{2}$.

<u>Alternative Solution</u>

By the Cauchy-Schwarz inequality, we have

$$\left(\frac{a}{b+c} + \frac{b}{a+c} + \frac{c}{a+b}\right)\left[a(b+c) + b(c+a) + c(a+b)\right] \geq (a+b+c)^2.$$

Hence it suffices to prove that

$$\frac{(a+b+c)^2}{a(b+c) + b(c+a) + c(a+b)} \geq \frac{3}{2}.$$

By the AM-GM inequality,

$$\begin{aligned}
2(a^2+b^2+c^2) &= 2a^2 + 2b^2 + 2c^2 + 4ab + 4bc + 4ca \\
&= (a^2+b^2) + (b^2+c^2) + (c^2+a^2) + 4ab + 4bc + 4ca \\
&\geq 2ab + 2bc + 2ca + 4ab + 4bc + 4ca \\
&= 3\left[a(b+c) + b(c+a) + c(a+b)\right]
\end{aligned}$$

so that the desired inequality follows.

5. Let $b_1 < b_2 < \cdots < b_n$ be a permutation of a_1, a_2, …, a_n in ascending order.

Since a_1, a_2, …, a_n are distinct positive integers, we have $b_i \geq i$ for all *i*.

Using the fact that $1 \geq \frac{1}{2^2} \geq \cdots \geq \frac{1}{n^2}$ and that Random Sum $\geq$ Reverse Sum, we have

$$\begin{aligned} a_1 + \frac{a_2}{2^2} + \cdots + \frac{a_n}{n^2} &\geq b_1 + \frac{b_2}{2^2} + \cdots + \frac{b_n}{n^2} \\ &\geq 1 + \frac{2}{2^2} + \cdots + \frac{n}{n^2} \\ &= 1 + \frac{1}{2} + \cdots + \frac{1}{n} \end{aligned}$$

Inequalities (Unit 2)

1. Setting $z = x + y$ and taking cube root on both sides, the original inequality becomes

$$x^2 + y^2 + (x+y)^2 \ge 3 \cdot \sqrt[3]{2} x^{\frac{2}{3}} y^{\frac{2}{3}} (x+y)^{\frac{2}{3}}.$$

Now, using the AM-GM inequality twice, we have

$$\begin{aligned} x^2 + y^2 + (x+y)^2 &= \frac{x^2+y^2}{2} + \frac{x^2+y^2}{2} + (x+y)^2 \\ &\ge \frac{x^2+y^2}{2} + \frac{2xy}{2} + (x+y)^2 \\ &= \frac{3}{2}(x+y)^2 \\ &= \frac{3}{2}(x+y)^{\frac{4}{3}} (x+y)^{\frac{2}{3}} \\ &\ge \frac{3}{2}\left(2\sqrt{xy}\right)^{\frac{4}{3}} (x+y)^{\frac{2}{3}} \\ &= 3 \cdot \sqrt[3]{2} x^{\frac{2}{3}} y^{\frac{2}{3}} (x+y)^{\frac{2}{3}} \end{aligned}$$

Hence the original inequality is proved.

2. Without loss of generality, assume $x \ge y \ge z$. Let $z = \frac{1}{3} - k$. Then $0 \le k \le \frac{1}{3}$.

Using the facts that $x + y = \frac{2}{3} + k$ and $xy \le \left(\frac{x+y}{2}\right)^2 = \left(\frac{1}{3} + \frac{k}{2}\right)^2$, we get

$$\begin{aligned} xy + yz + zx - 3xyz &\le z(x+y) + \left(\frac{x+y}{2}\right)^2 (1-3z) \\ &= \left(\frac{1}{3} - k\right)\left(\frac{2}{3} + k\right) + \left(\frac{1}{3} + \frac{k}{2}\right)^2 (3k) \\ &= \frac{2}{9} + \frac{3}{4}k^3 \\ &\le \frac{2}{9} + \frac{3}{4}\left(\frac{1}{3}\right)^3 \\ &\le \frac{1}{4} \end{aligned}$$

<u>Alternative Solution</u>

Note that

$$\begin{aligned} xy+yz+zx-3xyz &= xy(1-z)+yz(1-x)+xz(1-y) \\ &= xy(x+y)+yz(y+z)+xz(z+x) \\ &= x^2(y+z)+y^2(x+z)+z^2(y+x) \\ &= x^2(1-x)+y^2(1-y)+z^2(1-z) \end{aligned}$$

Since $x^2(1-x)-\frac{x}{4}=-\frac{x}{4}(1-2x)^2\le 0$, we have $x^2(1-x)\le\frac{x}{4}$.

Similarly, $y^2(1-y)\le\frac{y}{4}$ and $z^2(1-z)\le\frac{z}{4}$. Consequently,

$$xy+yz+zx-3xyz=x^2(1-x)+y^2(1-y)+z^2(1-z)\le\frac{x}{4}+\frac{y}{4}+\frac{z}{4}=\frac{1}{4}.$$

3. After some trial, we find that equality holds when the two triangles on the left hand side are similar, i.e. when

$$\frac{x}{x'}=\frac{y}{y'}=\frac{z}{z'}.$$

This is clearly the equality condition for the Cauchy-Schwarz inequality. Therefore, we attempt to use the Cauchy-Schwarz inequality to solve the problem.

Now we must express the area of a triangle in terms of its side lengths. Clearly, we should use the Heron's formula, which states that the area of a triangle with side lengths *a*, *b*, *c* is

$$\sqrt{s(s-a)(s-b)(s-c)},$$

where $s=\frac{a+b+c}{2}$. Hence the original inequality becomes

$$\begin{aligned} &\sqrt[4]{s(s-x)(s-y)(s-z)}+\sqrt[4]{s'(s'-x')(s'-y')(s'-z')} \\ &\qquad\le\sqrt[4]{(s+s')(s+s'-x-x')(s+s'-y-y')(s+s'-z-z')} \end{aligned}$$

with $s=\frac{x+y+z}{2}$ and $s'=\frac{x'+y'+z'}{2}$. Now, using the Cauchy-Schwarz inequality twice, we have

$$\begin{aligned} &\sqrt[4]{s(s-x)(s-y)(s-z)}+\sqrt[4]{s'(s'-x')(s'-y')(s'-z')} \\ &\le\sqrt{\left[\sqrt{s(s-x)}+\sqrt{s'(s'-x')}\right]\cdot\left[\sqrt{(s-y)(s-z)}+\sqrt{(s'-y')(s'-z')}\right]} \\ &\le\sqrt{\sqrt{(s+s')(s-x+s'-x')}\cdot\sqrt{(s-y+s'-y')(s-z+s'-z')}} \\ &=\sqrt[4]{(s+s')(s+s'-x-x')(s+s'-y-y')(s+s'-z-z')} \end{aligned}$$

and so the original inequality is proved.

4. Let $a_{2004} = 1 - a_1 - a_2 - \cdots - a_{2003}$. Then $a_1 + a_2 + \cdots + a_{2004} = 1$ and

$$\frac{a_1 a_2 \cdots a_{2003}(1 - a_1 - a_2 - \cdots - a_{2003})}{(a_1 + a_2 + \cdots + a_{2003})(1-a_1)(1-a_2)\cdots(1-a_{2003})} = \frac{a_1 a_2 \cdots a_{2004}}{(1-a_1)(1-a_2)\cdots(1-a_{2004})}.$$

By the AM-GM inequality,

$$\begin{aligned}
&(1-a_1)(1-a_2)\cdots(1-a_{2004}) \\
&= (a_2 + a_3 + \cdots + a_{2004})(a_1 + a_3 + \cdots + a_{2004})\cdots(a_1 + a_2 + \cdots + a_{2003}) \\
&\geq \left(2003 \cdot \sqrt[2003]{a_2 a_3 \cdots a_{2004}}\right)\left(2003 \cdot \sqrt[2003]{a_1 a_3 \cdots a_{2004}}\right)\cdots\left(2003 \cdot \sqrt[2003]{a_1 a_2 \cdots a_{2003}}\right) \\
&= 2003^{2004} a_1 a_2 \cdots a_{2004}
\end{aligned}$$

Hence we have $\dfrac{a_1 a_2 \cdots a_{2004}}{(1-a_1)(1-a_2)\cdots(1-a_{2004})} \leq \dfrac{1}{2003^{2004}}$.

Furthermore, equality holds when $a_1 = a_2 = \cdots = a_{2004} = \dfrac{1}{2004}$.

Therefore, the answer is $\dfrac{1}{2003^{2004}}$.

5. By the Cauchy-Schwarz inequality, $\displaystyle\sum_{k=1}^{n}\left(\frac{a_k{}^2}{a_k + b_k}\right) \cdot \sum_{k=1}^{n}(a_k + b_k) \geq \left(\sum_{k=1}^{n} a_k\right)^2$.

Hence $\displaystyle\sum_{k=1}^{n}\left(\frac{a_k{}^2}{a_k + b_k}\right) \geq \frac{\left(\sum_{k=1}^{n} a_k\right)^2}{\sum_{k=1}^{n}(a_k + b_k)} = \frac{\left(\sum_{k=1}^{n} a_k\right)^2}{2 \cdot \sum_{k=1}^{n} a_k} = \sum_{k=1}^{n}\left(\frac{a_k}{2}\right)$.

<u>Alternative Solution</u>

For real numbers *a* and *b*, we have $(a+b)^2 \geq \left(2\sqrt{ab}\right)^2 = 4ab$, so $\dfrac{ab}{a+b} \leq \dfrac{a+b}{4}$. Hence

$$\sum_{k=1}^{n}\left(\frac{a_k^{\ 2}}{a_k+b_k}\right)=\sum_{k=1}^{n}\left(\frac{a_k^{\ 2}+a_kb_k-a_kb_k}{a_k+b_k}\right)$$
$$=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}\left(\frac{a_kb_k}{a_k+b_k}\right)$$
$$\geq\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}\left(\frac{a_k+b_k}{4}\right)$$
$$=\sum_{k=1}^{n}a_k-\sum_{k=1}^{n}\left(\frac{2a_k}{4}\right)$$
$$=\sum_{k=1}^{n}\left(\frac{a_k}{2}\right)$$

6. Let $x=a+b-c$, $y=b+c-a$ and $z=c+a-b$. Then the original inequality becomes

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\leq\sqrt{\frac{x+y}{2}}+\sqrt{\frac{y+z}{2}}+\sqrt{\frac{z+x}{2}}.$$

By the AM-GM inequality, we have

$$\left(\frac{\sqrt{x}+\sqrt{y}}{2}\right)^2=\frac{x+y+2\sqrt{xy}}{4}=\frac{x+y}{4}+\frac{\sqrt{xy}}{2}\leq\frac{x+y}{4}+\frac{x+y}{4}=\frac{x+y}{2}$$

and hence $\frac{\sqrt{x}+\sqrt{y}}{2}\leq\sqrt{\frac{x+y}{2}}$. Similarly, we have

$$\frac{\sqrt{y}+\sqrt{z}}{2}\leq\sqrt{\frac{y+z}{2}}\ \text{ and }\ \frac{\sqrt{z}+\sqrt{x}}{2}\leq\sqrt{\frac{z+x}{2}}.$$

Adding these three inequalities, we have

$$\sqrt{x}+\sqrt{y}+\sqrt{z}\leq\sqrt{\frac{x+y}{2}}+\sqrt{\frac{y+z}{2}}+\sqrt{\frac{z+x}{2}},$$

thereby proving the original inequality.

Finally, equality in the above application of AM-GM inequality occurs if $\sqrt{x}=\sqrt{y}$, i.e. $x=y$. Similarly we must have $y=z$ and $z=x$. If $x=y$, then $a+b-c=b+c-a$, hence $2a=2c$ and $a=c$. Similarly, we must have $a=b$ and $b=c$. That is, equality holds if and only if $a=b=c$.

7. By the Cauchy-Schwarz inequality, we have

$$\left(x^2+y^2+z^2\right)\left(1^2+1^2+1^2\right)\geq(x+y+z)^2$$

which gives $x+y+z\le\sqrt{3\left(x^2+y^2+z^2\right)}$. On the other hand, the AM-GM inequality asserts that

$$xy+yz+zx\ge 3(xyz)^{\frac{2}{3}} \text{ and } \sqrt{x^2+y^2+z^2}\ge\sqrt{3}(xyz)^{\frac{1}{3}}.$$

Consequently, we have

$$\begin{aligned}
\frac{xyz\left(x+y+z+\sqrt{x^2+y^2+z^2}\right)}{\left(x^2+y^2+z^2\right)(xy+yz+zx)} &\le \frac{xyz\left(\sqrt{3\left(x^2+y^2+z^2\right)}+\sqrt{x^2+y^2+z^2}\right)}{\left(x^2+y^2+z^2\right)(xy+yz+zx)}\\
&= \frac{xyz\left(\sqrt{3}+1\right)\left(\sqrt{x^2+y^2+z^2}\right)}{\left(x^2+y^2+z^2\right)(xy+yz+zx)}\\
&= \frac{\left(\sqrt{3}+1\right)xyz}{\left(\sqrt{x^2+y^2+z^2}\right)(xy+yz+zx)}\\
&\le \frac{\left(\sqrt{3}+1\right)xyz}{\sqrt{3}(xyz)^{\frac{1}{3}}\cdot 3(xyz)^{\frac{2}{3}}}\\
&= \frac{\sqrt{3}+1}{3\sqrt{3}}\\
&= \frac{3+\sqrt{3}}{9}
\end{aligned}$$

8. By the rearrangement inequality, we have $a^3+b^3\ge a^2b+ab^2=ab\left(a+b\right)$.

   Similarly, we have $b^3+c^3\ge bc\left(b+c\right)$ and $c^3+a^3\ge ca\left(c+a\right)$. Consequently,

$$\begin{aligned}
&\frac{1}{a^3+b^3+abc}+\frac{1}{b^3+c^3+abc}+\frac{1}{c^3+a^3+abc}\\
&\le \frac{1}{ab\left(a+b\right)+abc}+\frac{1}{bc\left(b+c\right)+abc}+\frac{1}{ca\left(c+a\right)+abc}\\
&= \frac{1}{ab\left(a+b+c\right)}+\frac{1}{bc\left(a+b+c\right)}+\frac{1}{ca\left(a+b+c\right)}\\
&= \frac{1}{abc}
\end{aligned}$$

9. Let $a-1=x^2$, $b-1=y^2$ and $c-1=z^2$ for some positive non-negative x, y, z. Then the original inequality becomes

$$x+y+z\le\sqrt{\left(z^2+1\right)\left[\left(x^2+1\right)\left(y^2+1\right)+1\right]}.$$

By the Cauchy-Schwarz inequality, we have

$$x+y=x\cdot 1+1\cdot y\le\sqrt{(x^2+1)(y^2+1)}\ .$$

Similarly, we have

$$x+y+z\le\sqrt{\left[(x+y)^2+1\right](z^2+1)}\le\sqrt{\left[\left(x^2+1\right)\left(y^2+1\right)+1\right]\left(z^2+1\right)}$$

and proof is complete.

10. (a) When n = 2 and $x_1=x_2=1$, $C\ge\dfrac{1(1)(1^2+1^2)}{(1+1)^4}=\dfrac{1}{8}$.

On the other hand, when $C=\dfrac{1}{8}$, the inequality holds for all real numbers $x_1,\ \ldots,\ x_n\ge 0$ since

$$\begin{aligned}\frac{1}{8}\left(\sum_{1\le i\le n}x_i\right)^4&=\frac{1}{8}\left[\left(\sum_{1\le i\le n}x_i\right)^2\right]^2\\&=\frac{1}{8}\left[\left(\sum_{1\le i\le n}x_i^{\ 2}\right)+2\left(\sum_{1\le i<j\le n}x_ix_j\right)\right]^2\\&\ge\frac{1}{8}\left[2\sqrt{2\left(\sum_{1\le i<j\le n}x_ix_j\right)\left(\sum_{1\le i\le n}x_i^{\ 2}\right)}\right]^2\qquad\text{(AM-GM inequality)}\\&=\left(\sum_{1\le i<j\le n}x_ix_j\right)\left(\sum_{1\le i\le n}x_i^{\ 2}\right)\\&=\sum_{1\le i<j\le n}x_ix_j(x_1^{\ 2}+x_2^{\ 2}+\cdots+x_n^{\ 2})\\&\ge\sum_{1\le i<j\le n}x_ix_j(x_i^{\ 2}+x_j^{\ 2})\end{aligned}$$

Hence the required least constant C is $\dfrac{1}{8}$.

(b) Consider the term with i = 1 and j = 2 in the last two expressions in (a).

We have $x_1x_2(x_1^{\ 2}+x_2^{\ 2}+\cdots+x_n^{\ 2})\ge x_1x_2(x_1^{\ 2}+x_2^{\ 2})$.

This equality holds if and only if $x_3=x_4=\cdots=x_n$.

As the choice of i and j is arbitrary, if any (n – 2) of the x_i ’s are zero, then equality in the last inequality holds, and vice versa

When (n – 2) of the x_i ’s are zero, the inequality is reduced to the case of n = 2.

Consider the application of AM-GM inequality in (a).

Equality holds if and only if $x_1{}^2 + x_2{}^2 = 2x_1x_2$, or $(x_1 - x_2)^2 = 0$, i.e. $x_1 = x_2$.

Hence equality of the original inequality holds if and only if any (n – 2) of the x_i's are zero and the remaining two x_i's are equal (possibly to zero).

Inequalities (Unit 3)

1. By Heron's formula, $T=\sqrt{\left(\frac{a+b+c}{2}\right)\left(\frac{-a+b+c}{2}\right)\left(\frac{a-b+c}{2}\right)\left(\frac{a+b-c}{2}\right)}$.

Putting this into the original inequality, the original inequality can be simplified as follows:

$$
\begin{aligned}
a^2+b^2+c^2 &\geq \sqrt{3(a+b+c)(-a+b+c)(a-b+c)(a+b-c)} \\
a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 &\geq 3\left[(b+c)^2-a^2\right]\left[a^2-(b-c)^2\right] \\
a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 &\geq 3\left[2bc+(b^2+c^2-a^2)\right]\left[2bc-(b^2+c^2-a^2)\right] \\
a^4+b^4+c^4+2a^2b^2+2b^2c^2+2c^2a^2 &\geq 3\left[2a^2b^2+2b^2c^2+2c^2a^2-a^4-b^4-c^4\right] \\
4a^4+4b^4+4c^4 &\geq 4a^2b^2+4b^2c^2+4c^2a^2
\end{aligned}
$$

By the AM-GM inequality, we have

$$
\begin{aligned}
4a^4+4b^4+4c^4 &= (2a^4+2b^4)+(2b^4+2c^4)+(2c^4+2a^4) \\
&\geq \left(2\sqrt{2a^4\cdot 2b^4}\right)+\left(2\sqrt{2b^4\cdot 2c^4}\right)+\left(2\sqrt{2c^4\cdot 2a^4}\right) \\
&= 4a^2b^2+4b^2c^2+4c^2a^2
\end{aligned}
$$

thereby proving the last inequality and hence the original inequality. It is clear in the application of the AM-GM inequality that equality holds if and only if $a=b=c$.

Alternative Solution

Without loss of generality, assume that the angle opposite the side a is acute. Suppose that the altitude from this vertex, whose length we denote by h, is of distances m and n from the remaining 2 vertices, with $b=\sqrt{h^2+m^2}$ and $c=\sqrt{h^2+n^2}$. WLOG, assume that $m\geq n$. Then $a=m+n$ or $m-n$.

For $a=m+n$, the original inequality becomes

$$(m+n)^2+(h^2+m^2)+(h^2+n^2)\geq 4\sqrt{3}\cdot\frac{(m+n)h}{2}.$$

Rewriting this as a quadratic equality in h, we have

$$h^2-\sqrt{3}(m+n)h+(m^2+mn+n^2)\geq 0.$$

The discriminant of the quadratic function on the left is

$$\Delta=\left[\sqrt{3}(m+n)\right]^2-4(1)(m^2+mn+n^2)=-(m-n)^2\leq 0.$$

Since the coefficient of h^2 is positive, this means $h^2-\sqrt{3}(m+n)h+(m^2+mn+n^2)\leq 0$ for all h, as desired. Equality holds when $m=n$, which means $b=c$. By symmetry, we need $a=b=c$.

For $a=m-n$, the argument is the same as above, with n replaced by $-n$.

2. Rewrite the given inequality as $c^2 = a^2 + b^2 - 2ab\cos 60°$.

Hence we see that *a*, *b*, *c* are the side lengths of a triangle where the angle opposite the side with length *c* is equal to 60°.

In a triangle, a side opposite a larger angle is longer. Since 60° = 180° ÷ 3, one other angle of the triangle must be at least 60° and the remaining angle must be at most 60°. In other words, if we assume (without loss of generality) that $a \geq b$, then we must have $a \geq c$ and $b \leq c$.

From this, we see that $a - c$ is positive while $b - c$ is negative, so that $(a-c)(b-c) \leq 0$.

3. By the Cauchy-Schwarz inequality, we have

$$\left(\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF}\right)\left(BC \cdot PD + CA \cdot PE + AB \cdot PF\right) \geq \left(BC + CA + AB\right)^2.$$

Hence

$$\frac{BC}{PD} + \frac{CA}{PE} + \frac{AB}{PF} \geq \frac{(BC + CA + AB)^2}{BC \cdot PD + CA \cdot PE + AB \cdot PF}.$$

The right hand side of the above inequality is a constant, since the numerator is the square of the perimeter while the denominator is twice the area.

Equality holds if and only if

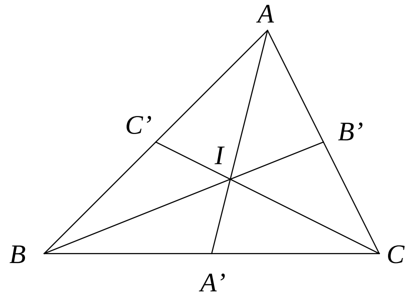
$$\frac{BC}{PD} : \frac{CA}{PE} : \frac{AB}{PF} = (BC \cdot PD) : (CA \cdot PE) : (AB \cdot PF),$$

or $PD = PE = PF$. In other words, the expression in the question is minimum when (and only when) *P* is the incentre of ΔABC.

4. Let $x = \dfrac{AI}{AA'}$, $y = \dfrac{BI}{BB'}$ and $z = \dfrac{CI}{CC'}$. The inequality to be proved is then $\dfrac{1}{4} \leq xyz \leq \dfrac{8}{27}$.

Note that

$$\begin{aligned}
& x + y + z \\
&= \frac{AI}{AP} + \frac{BI}{BQ} + \frac{CI}{CR} \\
&= \frac{[ABI] + [CAI]}{[ABC]} + \frac{[BAI] + [BCI]}{[ABC]} + \frac{[CAI] + [CBI]}{[ABC]} \\
&= \frac{2\left([ABI] + [BCI] + [CAI]\right)}{[ABC]} \\
&= \frac{2[ABC]}{[ABC]} \\
&= 2
\end{aligned}$$

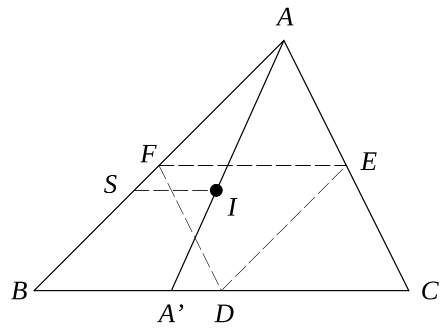


Hence the AM-GM inequality asserts that

$$xyz \le \left(\frac{x+y+z}{3}\right)^3 = \left(\frac{2}{3}\right)^3 = \frac{8}{27},$$

thereby proving the right-hand inequality.

To prove the left-hand inequality, we first make some additional observation as follows. Let *D*, *E*, *F* be the mid-points of *BC*, *CA*, *AB* respectively. We claim that *I* lies in Δ*DEF*. Assuming the claim, we draw a line through *I* parallel to *BC* cutting *AB* at *S*. Since Δ*ASI* ~ Δ*ABA'*, we have



$$x = \frac{AI}{AA'} = \frac{AS}{AB} > \frac{AF}{AB} = \frac{1}{2}.$$

Similarly, we have $y > \frac{1}{2}$ and $z > \frac{1}{2}$.

Now we return to the proof of the claim, namely, that *I* lies in Δ*DEF*. Indeed, the angle bisector theorem yields

$$\frac{AI}{IA'} = \frac{AB}{BA'} \text{ and } \frac{BA'}{A'C} = \frac{AB}{AC},$$

so that $\frac{BA'}{BC} = \frac{AB}{AB+AC}$ and hence $\frac{AI}{IA'} = \frac{AB+AC}{BC} > \frac{BC}{BC} = 1$ by the triangle inequality. Hence *I* is ‘below’ *EF* in the figure, and the same is true with respect to *DF* and *DE*, thereby establishing the claim.

Consequently, we may write

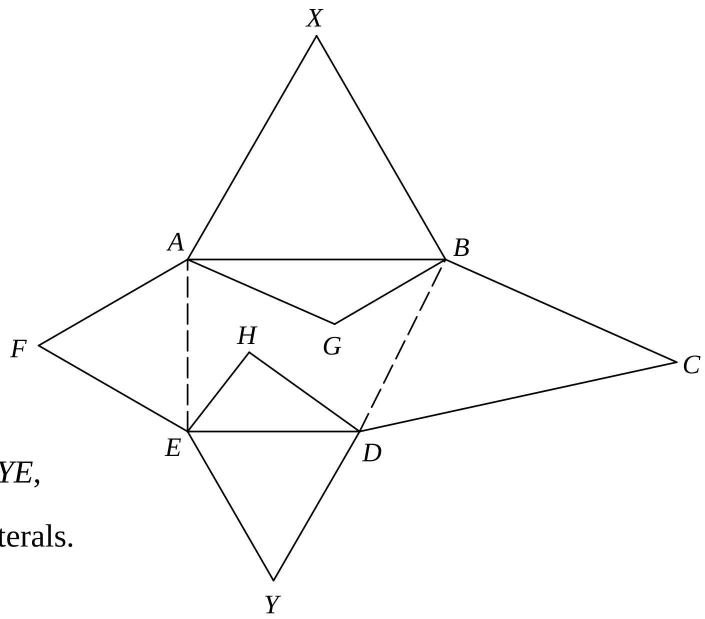
$$x = \frac{1}{2} + \alpha,\ y = \frac{1}{2} + \beta \text{ and } z = \frac{1}{2} + \gamma$$

for some $\alpha, \beta, \gamma > 0$. Then we have

$$\begin{aligned} xyz &= \left(\frac{1}{2}+\alpha\right)\left(\frac{1}{2}+\beta\right)\left(\frac{1}{2}+\gamma\right) \\ &= \frac{1}{8} + \frac{1}{4}(\alpha+\beta+\gamma) + \frac{1}{2}(\alpha\beta+\beta\gamma+\gamma\alpha) + \alpha\beta\gamma \\ &> \frac{1}{8} + \frac{1}{4}(\alpha+\beta+\gamma) \\ &= \frac{1}{8} + \frac{1}{4}\left(\frac{1}{2}\right) \\ &= \frac{1}{4} \end{aligned}$$

and hence proving the left-hand inequality.

5. Let $AB = BC = CD = a$ and $DE = EF = FA = b$. As shown in the figure, construct equilateral triangles ABX and DEY. Since $\angle AXB = \angle DYE = 60°$, $AX = XB = BD = a$ and $DY = YE = EA = b$, the two hexagons $ABCDEF$ and $AXBDYE$ are congruent and so $CF = XY$.



Since

$$\angle AXB + \angle AGB = 180° = \angle DHE + \angle DYE,$$

$AXBG$ and $DYEH$ are cyclic quadrilaterals. Hence by the Ptolemy's theorem,

$$AB \cdot XG = AX \cdot BG + XB \cdot AG,$$

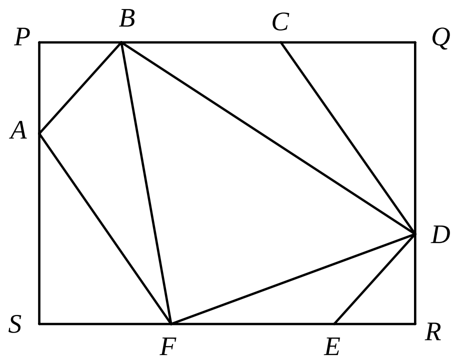
which is equivalent to $aXG = aBG + aAG$, or

$$XG = BG + AG.$$

Similarly, we have $YH = DH + EH$ and hence

$$AG + GB + GH + DH + HE = XG + GH + YH \geq XY = EF.$$

6. As shown in the figure, extend BC and EF to draw a rectangle $PQRS$ enclosing the hexagon. Since opposite sides of the hexagon are parallel, opposite angles are equal (i.e. $\angle A = \angle D$, $\angle B = \angle E$ and $\angle C = \angle F$). Let a, b, c, d, e, f denote the lengths of AB, BC, CD, DE, EF and FA respectively. We have



$$\begin{aligned} 2BF &\geq PA + AS + QD + DR \\ &= a\sin B + f\sin F + c\sin C + d\sin E \\ &= a\sin B + f\sin C + c\sin C + d\sin B \end{aligned}$$

Hence

$$R_A = \frac{BF}{2\sin A} = \frac{1}{4}\left(\frac{a\sin B}{\sin A} + \frac{f\sin C}{\sin A} + \frac{c\sin C}{\sin A} + \frac{d\sin B}{\sin A}\right).$$

Similarly, we have

$$R_C = \frac{1}{4}\left(\frac{c\sin A}{\sin C} + \frac{b\sin B}{\sin C} + \frac{e\sin B}{\sin C} + \frac{f\sin A}{\sin C}\right)$$

and

$$R_E = \frac{1}{4}\left(\frac{e\sin C}{\sin B} + \frac{d\sin A}{\sin B} + \frac{a\sin A}{\sin B} + \frac{b\sin C}{\sin B}\right).$$

Adding these inequalities and using the fact that $\frac{x}{y} + \frac{y}{x} \geq 2\sqrt{\frac{x}{y}\cdot\frac{y}{x}} = 2$ for $x, y > 0$, we have

$$R_A + R_C + R_E \geq \frac{1}{4}(2a + 2b + 2c + 2d + 2e + 2f) = \frac{p}{2}$$

as desired.